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The effect of a non-linear basic temperature profile on the forced flow of a viscous liquid non-uniformly heated from below

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The forced motion of a liquid heated non-uniformly from below and subject to a non-constant basic temperature profile is examined. The problem is formulated as a two-point boundary-value problem which is solved by a numerical method. In agreement with previous work, the presence of a positive vertical lapse rate of basic temperature tends to decrease the forced motion. It is found that constant heat generation within the fluid produces a parabolic basic temperature profile, and this tends to increase the forced velocity components.

1. Statement of the problem

Many authors have discussed both experimentally and theoretically the motion of a viscous fluid which is differentially heated from below. From experimental work (see, for example, Fultz 1956) two distinct régimes have emerged. At low Rossby numbers a wave régime is established, and at high Rossby numbers the motion is symmetrical (with the liquid rising at the heat source and being simultaneously deflected by rotation); this is known as the Hadley régime.

Davies (1953) and Kuo (1954) formulated theories for the symmetrical Hadley régime, restricting the results to the case of a shallow fluid with a basic temperature profile independent of depth. These results were generalized to a fluid of greater depth by Lance & Deland (1957), who solved the equations on a differential analyser. Lance (1958) also used a numerical method to include in the problem a constant positive vertical lapse rate of basic temperature.

The present investigation aims to generalize this model to include a variable vertical lapse rate, and to examine the influence of this on the symmetrical motion of the liquid when it is differentially heated at the base. This non-linear basic temperature profile arises from constant heating (or cooling) within the fluid. This may be due to radiative heat loss as incorporated by Ray & Scorer (1963), or to heating by a Joule electrolyte as envisaged by Sparrow, Goldstein & Jonsson (1964*a*). These two investigations are concerned with the onset of convective instability, and they show that the presence of the non-linear basic temperature profile tends to destabilize the fluid.

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2. The equations of motion

A simple model is constructed in which an incompressible viscous fluid occupies a cylindrical dishpan, and the problem is formulated in cylindrical polar co-ordinates (r, ϕ, z). The motion is assumed to be steady and axially symmetric, a constant heating term is included, and the Coriolis force is put equal to zero. If rotation is included in this problem the following method of analysis is still applicable; but, by comparison with Davies (1953), it is expected that in general the magnitudes of the radial and vertical velocity components would be decreased by increasing rotation.

Under these assumptions the set of equations connecting the dependent variables is

$$\mathbf{v}_1 \wedge \operatorname{curl} \mathbf{v}_1 = g\mathbf{k} + \rho_1^{-1} \operatorname{grad} p_1 + \operatorname{grad} \tfrac{1}{2}\mathbf{v}_1^2 + \nu \operatorname{curl} \operatorname{curl} \mathbf{v}_1, \tag{1}$$

$$\operatorname{div} \mathbf{v}_1 = 0, \tag{2}$$

$$\rho_1 - \rho_s = -\alpha(T_1 - T_s), \tag{3}$$

$$(\mathbf{v}_1 . \operatorname{grad}) T_1 = \kappa \nabla^2 T_1 + Q_0/\rho_1 c_v, \tag{4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

In these equations $\mathbf{v}_1$ is the velocity vector, the pressure is denoted by p_1, the density by ρ_1, the temperature by T_1, ν is the coefficient of kinematic viscosity, and κ is the coefficient of thermometric conductivity of the fluid. $\mathbf{k}$ is a unit vector in the z-direction, thus the gravitational acceleration g acts in this direction. Equation (3) is the equation of state for a liquid valid within small temperature ranges, where the reciprocal of the coefficient of cubical expansion α is taken as a constant; ρ_s and T_s are a constant density and temperature respectively. Equation (4) is the heat-transfer equation incorporating the constant heating term Q_0, and c_v is the specific heat at constant volume.

Initially, when the fluid is not differentially heated, it is assumed that no motion occurs. Therefore, denoting basic state quantities by suffix zero and assuming they are functions of z only, the basic state equations become

$$-\rho_0^{-1} dp_0/dz - g = 0, \tag{5}$$

$$\rho_0 - \rho_s = -\alpha(T_0 - T_s), \tag{6}$$

$$d^2T_0/dz^2 = -Q_0/\kappa\rho_s c_v. \tag{7}$$

In equation (7) the basic state density has been replaced by ρ_s under the Boussinesq approximation. This equation can be integrated twice to give

$$T_0 = -(Q_0 h^2/2\kappa\rho_s c_v)\bar{z}^2 + M\bar{z} + N, \tag{8}$$

where $\bar{z}$ is a non-dimensional height $\bar{z} = z/h$, h being the depth of fluid, and M, N are constants to be determined from the boundary conditions. To achieve this the temperatures of the two horizontal bounding surfaces of the fluid are assumed to be given by

$$\left.\begin{aligned} T_0 &= T_{(1)} \quad \text{at} \quad \bar{z} = 0, \\ T_0 &= T_{(2)} \quad \text{at} \quad \bar{z} = 1 (z = h). \end{aligned}\right\} \tag{9}$$

Denoting the temperature difference between the two boundaries as θh, the basic temperature profile can be written as

$$T_0 = (Q_0 h^2/2\kappa\rho_s c_v)(\bar{z}-\bar{z}^2)+\theta h\bar{z}+T_{(1)}. \tag{10}$$

When the liquid is differentially heated the symmetrical departure from this basic state is assumed to be small, and perturbation quantities u, v, w, p, ρ, τ are introduced. Here the perturbation velocity vector $\mathbf{v}$ has been written in its component form u, v, w, where u is in the r-increasing direction, v in the ϕ-increasing direction and w in the z-increasing direction. p, ρ, τ are the perturbation pressure, density and temperature respectively, thus the complete temperature is given by $T_0(z)+\tau(r,z)$. These quantities must satisfy the linearized perturbation equations of motion, which under the Boussinesq approximation become

$$-\partial p/\partial r+\nu\rho_s(\nabla^2 u-u/r^2) = 0, \tag{11}$$

$$\nabla^2 v - v/r^2 = 0, \tag{12}$$

$$-\partial p/\partial z+g\alpha\tau+\nu\rho_s\nabla^2 w = 0, \tag{13}$$

$$\partial(ru)/\partial r+\partial(rw)/\partial z = 0, \tag{14}$$

$$w\,dT_0/dz = \kappa\nabla^2\tau. \tag{15}$$

It should be noted that no perturbation heating term is included here.

3. Reduction to a two-point boundary-value problem

The variables r and z are separated following Davies (1953) by the substitutions

$$\left.\begin{aligned} u &= U(z)J_1(\beta r), \quad v = 0, \quad w = W(z)J_0(\beta r),\\ \tau &= T(z)J_0(\beta r), \quad p = P(z)J_0(\beta r), \end{aligned}\right\} \tag{16}$$

where J_0, J_1 are Bessel functions of orders 0 and 1 respectively. The set of partial differential equations now reduces to a system of ordinary differential equations in which the variables are functions of z only. Eliminating P, U and T, and using equation (10) yields the following equations for the vertical velocity component W:

$$(D^2-a^2)^3 W = a^2\{Q(1-2\bar{z})+A\}W, \tag{17}$$

where $a = \beta h$, $D = d/d\bar{z}$. A and Q are non-dimensional parameters defined as

$$A = h^4 g\alpha\theta/\kappa\nu\rho_s, \quad Q = Q_0 h^5 g\alpha/2\kappa^2\nu\rho_s^2 c_v, \tag{18}$$

A is the Rayleigh number and Q is a radiational parameter as defined by Ray & Scorer (1963). The dimensionless ratio $2Q/A = hQ_0/\theta\kappa\rho_s c_v$ is the ratio of the total flux of heat due to body heating to the flux due to conduction through the top and bottom boundaries under the same temperature difference but without body heating. A similar parameter is defined by Sparrow *et al.* (1964*a*) and by other authors in problems where heat transfer by radiative and conductive processes is considered.

The six boundary conditions for the perturbed state must now be formulated; these are unaltered by the inclusion of internal heating and are identical with those formulated by Davies (1953). The fluid is assumed to be bounded above at

$z = h$ by a free surface, and below at $z = 0$ by a rigid surface; four of the boundary conditions concern the velocities at these two surfaces, namely

$$w = 0, \quad \partial u/\partial z = 0 \quad \text{at} \quad z = h, \tag{19}$$

$$w = 0, \quad u = 0 \quad \text{at} \quad z = 0. \tag{20}$$

In addition to these conditions a side-wall condition must be satisfied, the radial velocity component must vanish on the circumference, that is

$$u = 0 \quad \text{at} \quad r = r_0,$$

where r_0 is the radius of the dishpan. Here the boundary layer at the side wall has been neglected; u is strictly zero at the inside edge of this boundary layer. From (16) this condition defines an infinite set of β, the first of which is given by β_1, where

$$\beta_1 r_0 = 3{\cdot}83. \tag{21}$$

Following Davies (1953) the base of the fluid is assumed to be heated around the circumference and cooled at the centre; this heating is prescribed as

$$\partial \tau/\partial z = HJ_0(\beta r) \quad \text{at} \quad z = 0, \tag{22}$$

where H is a positive constant. This heating function implies that the net flow of heat by conduction at the base is zero. At the free surface it is postulated that there is a fixed heat flux due to conduction which is unchanged by the perturbation flow, thus

$$\partial \tau/\partial z = 0 \quad \text{at} \quad z = h. \tag{23}$$

The boundary conditions (19), (20), (22) and (23) can now be expressed entirely in terms of the dependent variable W. This variable is then non-dimensionalized by putting $W = \bar{K}\bar{W}$, where $\bar{K} = g\alpha H h^3/\rho_s \nu$ has the dimensions of velocity. The problem has now been reduced to a two-point boundary-value problem, and can be written in terms of $\bar{W}$ as

$$(D^2 - a^2)^3 \bar{W} = a^2\{Q(1-2\bar{z}) + A\}\bar{W}, \tag{24}$$

subject to

$$\left.\begin{aligned} \bar{W} &= 0, \quad D\bar{W} = 0, \quad D(D^2-a^2)^2\bar{W} = a^2, \quad \text{at} \quad \bar{z} = 0,\\ \bar{W} &= 0, \quad D^2\bar{W} = 0, \quad D(D^2-a^2)^2\bar{W} = 0, \quad \text{at} \quad \bar{z} = 1. \end{aligned}\right\} \tag{25}$$

The dimensionless radial velocity component $\bar{U}$, defined as $U = \bar{K}\bar{U}$ can then be calculated from the equation of continuity (14) as

$$\bar{U} = -D\bar{W}/a. \tag{26}$$

It is interesting to note that if parameters T_a and μ are introduced into equation (24) by the transformations

$$(Q+A) = -T_a, \quad 2Q/(Q+A) = (1-\mu),$$

the resulting equation becomes

$$(D^2-a^2)^3\bar{W} = -T_a a^2\{1-(1-\mu)\bar{z}\}\bar{W}.$$

It has been pointed out by Dr M. H. Rogers in a private communication that this has the same form as the equation defining marginal stability for Couette flow between two cylinders under the small-gap approximation (see, for example, Chandrasekhar 1961). However, the boundary conditions in the two problems are different.

4. Numerical procedure

Because a term in $\bar{z}$ appears on the right-hand side of the differential equation an analytic solution has not been sought, but the problem has been solved by a numerical method. This is based on the fact that the differential equation (24) is linear, and can be split up into six first-order linear equations.

Difficulty lies in the fact that the boundary conditions are known at each end of the range; thus, if the integration is started at $\bar{z} = 1$, the values of $D\overline{W}(1)$, $D^3\overline{W}(1)$, $D^4\overline{W}(1)$ must be assumed. Three different sets of guesses are taken for these derivatives, and, using the same negative increment in each case, the equation is integrated by the Runge-Kutta procedure to the other end-point, $\bar{z} = 0$. Each set of guesses gives a different value of $\overline{W}$, say w_i $(i = 1, 2, 3)$ where w_i are linearly independent solutions. The true value of $\overline{W}$ must be a linear combination of w_i, say

$$\overline{W} = \sum_{i=1}^{3} A_i w_i,$$

where A_i are constants. These constants are taken such that $\overline{W}$ satisfies the three boundary conditions at $\bar{z} = 0$, namely

$$\overline{W}(0) = 0, \quad D\overline{W}(0) = 0, \quad D(D^2-a^2)^2\,\overline{W}(0) = a^2.$$

Numerically this is achieved by solving the three linear equations for A_i by inverting the matrix of coefficients. The value of $\overline{W}$ is then computed at points in the range, and $\overline{U}$ can be calculated from a knowledge of the first derivative of $\overline{W}$.

A similar method has been used by Harris & Reid (1964) and Sparrow, Munro & Jonsson (1964*b*) for solving linear eigenvalue problems. In these cases the determinant of coefficients is required to be zero, and an iterative scheme is set up.

5. Calculation of the velocity profiles

Taking $\beta = \beta_1$ the bounday condition (21) gives $ar_0 = 3{\cdot}83h$. In these calculations a is taken to be unity, thus the depth of the fluid is small compared with the radius of the cylindrical vessel. The same numerical procedure holds for all values of a, and is not restricted to the case of a shallow fluid. Lance & Deland (1957) give results for the problem with no body heating for $a = 1$, 4, 8. These show broadly that as the fluid depth is increased the magnitudes of the vertical and radial velocity components decrease.

Numerical values of the non-dimensional parameter A and Q defined in (18) are also required. Three separate cases are considered in the subsections to follow.

(i) $Q = 0, \quad A > 0$

First, the case is considered in which there is no internal heat generation, the basic temperature gradient then reduces to a constant. This is the problem considered by Lance (1958) and solved by the method described by Goodman & Lance (1956).

Lance considers a model in which a dishpan of radius 15 cm contains water, for which $\nu = 0{\cdot}01\ \mathrm{cm}^2/\mathrm{sec}$ and the Prandtl number $\sigma = \nu/\kappa = 7$ at 15 °C. As $a = 1$ this implies that the depth of water is 3·916 cm. Taking $\theta = 1\ ^\circ\mathrm{C/cm}$, Lance calculates $A = h^2\sigma\theta/\nu = 10{,}737$. However, A as defined by this relation is a dimensional parameter (deg. sec. cm^{-1}). This present investigation uses the non-dimensional definition of A given in (18) with the numerical values above and $\rho_s = 1$; this gives $A = h^4 g\alpha\theta/\kappa\nu\rho_s = 4{\cdot}13\times10^6$.

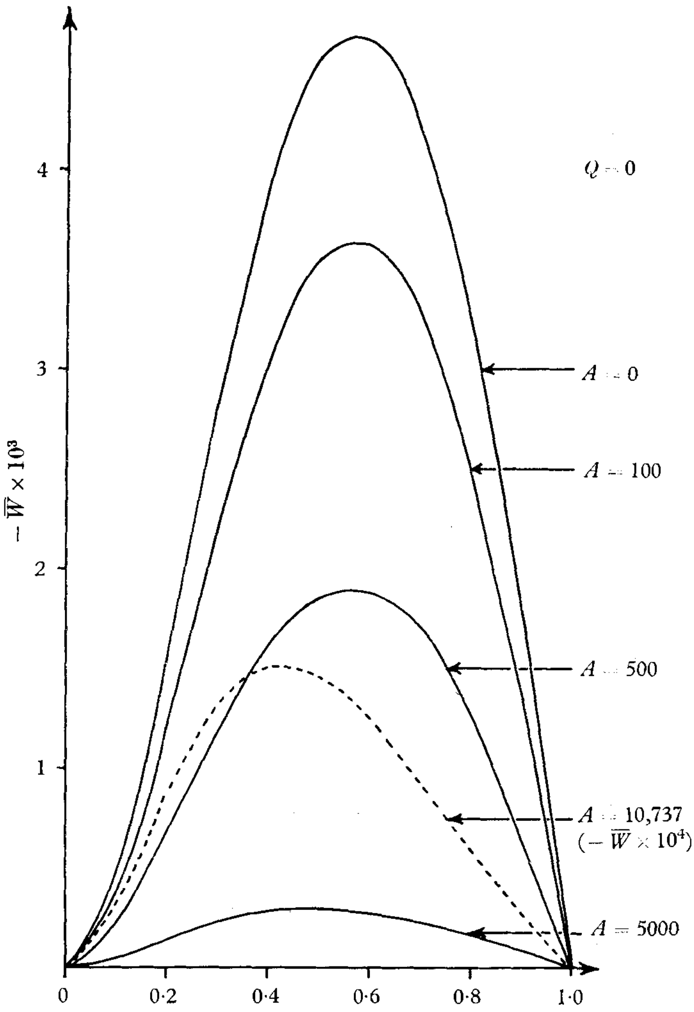

FIGURE 1. Non-dimensional vertical velocity profile plotted against non-dimensional height for $Q = 0$; $A \geqslant 0$.

The differential equation (24) subject to the boundary conditions (25) was solved by the numerical procedure outlined in § 4. A was taken to have values between 0 and 2×10^4, and in each case an increment of $-0{\cdot}1$ in the integration formula gave results accurate to three significant figures. For several values of A curves of $\overline{W}$ and $\overline{U}$ are plotted against $\bar{z}$ in figures 1 and 2 respectively.

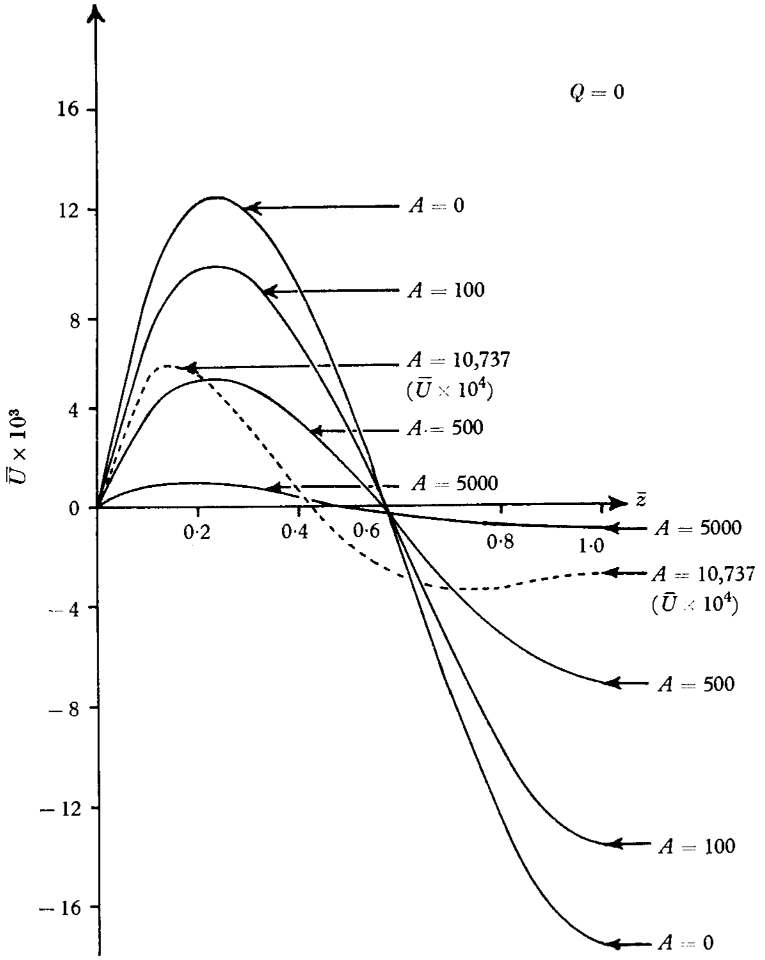

FIGURE 2. Non-dimensional radial velocity profile plotted against non-dimensional height for $Q = 0$; $A \geqslant 0$.

The values for an isothermal basic state, $A = 0$, agree with those of Lance & Deland (1957) in the case of zero Reynolds number. The case $A = 10{,}737$ agrees with Lance's calculation when the Reynolds number is zero, and intermediate cases also show that as the positive lapse rate is increased the magnitude of the forced velocity components is reduced, that is the stability is increased. The three other main features noted by Lance (1958) are also apparent here as A increases.

(ii) $Q > 0, \quad A = 0$

This second case considers a fluid in which the two boundary surfaces are maintained at the same basic temperature, but heat is generated internally within the fluid. The basic temperature is then given from (10) as

$$T_0 = (Q_0 h^2/2\kappa\rho_s c_v)(\bar{z} - \bar{z}^2) + T_{(1)},$$

where $T_{(1)}$ is the temperature at $\bar{z} = 0$ and $\bar{z} = 1$. The temperature attains a maximum value of $T_{(m)}$ at $\bar{z} = 0{\cdot}5$ where

$$T_{(m)} = (Q_0 h^2/8\kappa\rho_s c_v) + T_{(1)}.$$

The temperature profile is therefore parabolic with the statical density in the upper half of the fluid increasing with height. Instability is therefore expected to set in at a particular value of $T_{(m)} - T_{(1)}$, this is the problem considered by Sparrow *et al.* (1964*a*), who find that for a fluid contained between two isothermal rigid boundaries instability first occurs (in our notation) when $Q = 18662{\cdot}6$, $a = 4{\cdot}0$. The main interest in this calculation for $a = 1$ lies in those ranges of Q and A for which the flow is stable with no forcing for all values of a.

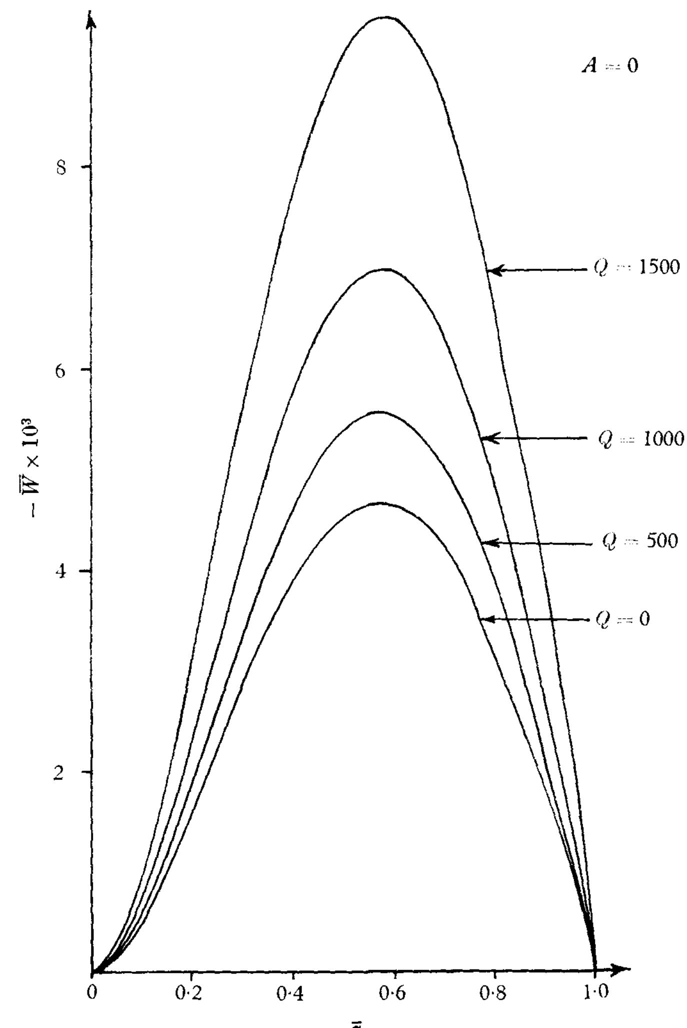

FIGURE 3. Non-dimensional vertical velocity profile plotted against non-dimensional height for $0 \leqslant Q \leqslant 1500$; $A = 0$.

The differential equation (24) with $a = 1, A = 0$ and various values of Q in the range of $0 \leqslant Q \leqslant 1500$ was solved numerically subject to the boundary conditions (25). Some results for $\overline{W}$ and $\overline{U}$ are shown in figures 3 and 4 respectively where the curves $Q = 0$ are included for completeness.

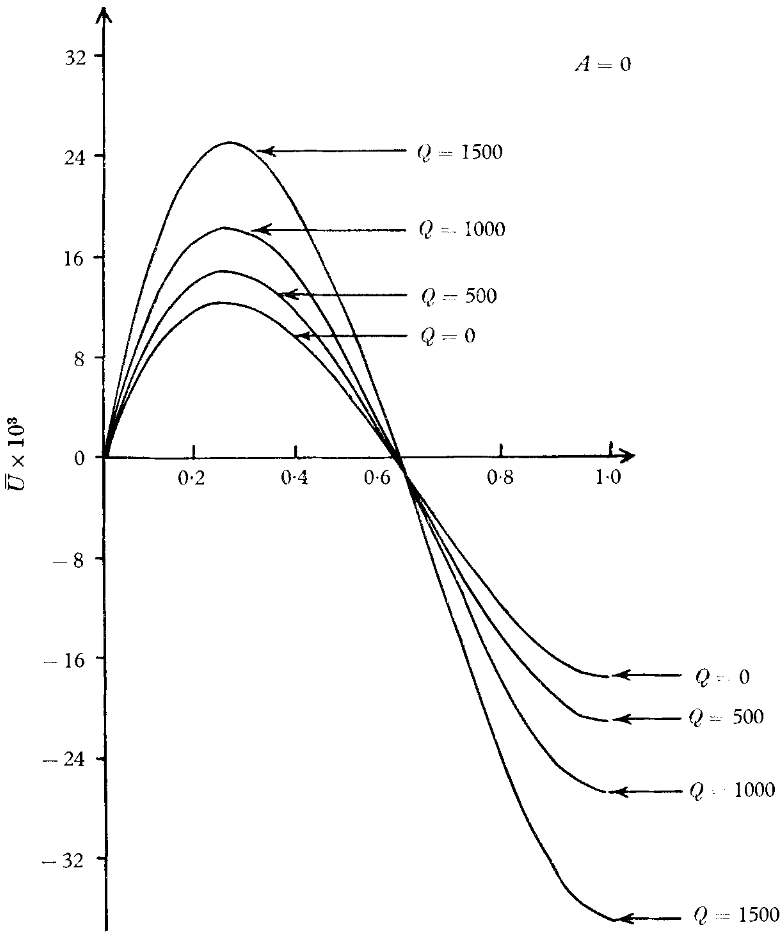

FIGURE 4. Non-dimensional radial velocity profile plotted against non-dimensional height for $0 \leqslant Q \leqslant 1500$; $A = 0$.

From these figures it is apparent that as Q is increased from zero in this range the magnitudes of the velocity components are increased. The position of the maximum vertical velocity component remains very nearly at $\bar{z} = 0{\cdot}57$ for all the values of Q. The radial velocity component $\overline{U}$ has a zero at this point, and the maximum inflow in all cases occurs at the free surface of the fluid.

As Q is increased further the velocity components continue to grow, and for Q of order 2×10^3 the linearized approximation is probably no longer valid. Presumably in an exact theory the velocity amplitudes would become infinite at some value of $Q = Q_s$ corresponding to actual neutral stability for $a = 1$. As the linearized approximation is built into this theory it is difficult to draw any definite conclusions about Q_s other than an order of magnitude at which instability sets in.

(iii) $Q > 0, \quad A > 0$

When both A and Q are positive constants the fluid has a higher basic temperature at its upper surface than at its lower surface, and there is also internal heat generation within the fluid.

From equation (10) the basic temperature profile is parabolic with a maximum value $T_{(m)}$ at $\bar{z} = 0{\cdot}5\,(1+A/Q)$, where

$$T_{(m)} - T_{(1)} = \theta^2\kappa\rho_s c_v/2Q_0 + \tfrac{1}{2}\theta h + Q_0 h^2/8\kappa\rho_s c_v.$$

For $A > Q$ the temperature therefore increases with height throughout the fluid, and, as noted by Sparrow *et al.* (1964*a*), this corresponds to a completely stable state. When $A < Q$ the temperature maximum lies within the fluid, and some of the fluid is at higher temperature than the upper bounding surface.

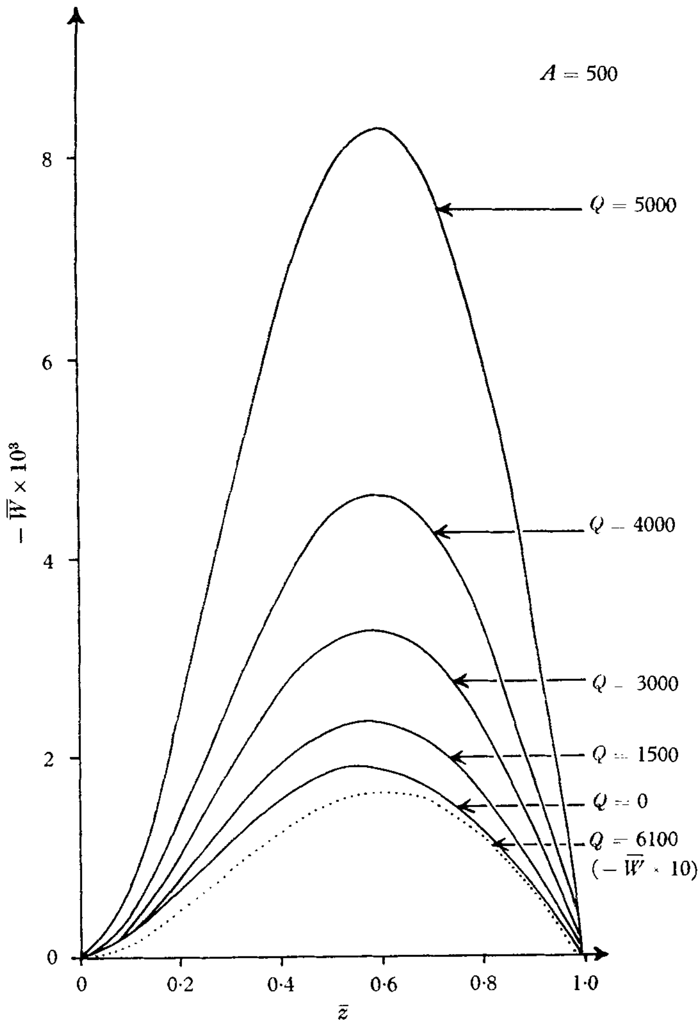

FIGURE 5. Non-dimensional vertical velocity profile plotted against non-dimensional height for $0 \leqslant Q \leqslant 6100$; $A = 500$.

The differential equation for $\overline{W}$ was solved, subject to the boundary conditions, for $A = 500$ and various values of Q; some of the results for $\overline{W}$ are shown in figure 5, and figure 6 gives corresponding results for $\overline{U}$. In the calculated results for Q in the range $0 \leqslant Q \leqslant 500$, that is $-1 \leqslant A/Q \leqslant \infty$, the magnitudes of the velocity components increase only very slightly as Q increases, because the basic temperature increases with height throughout the fluid, and for this reason are not shown.

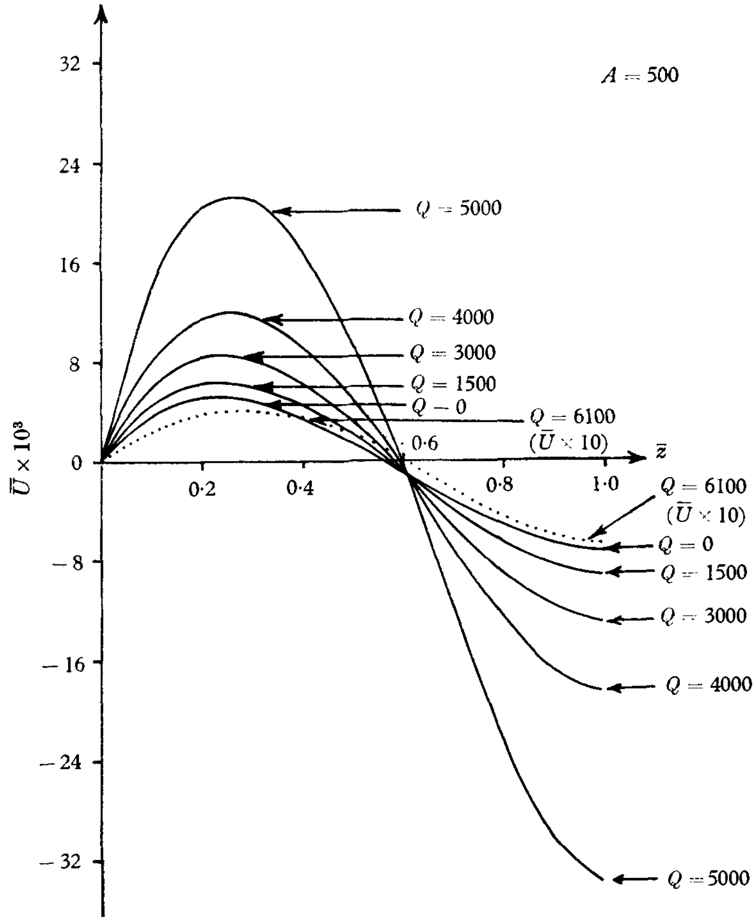

FIGURE 6. Non-dimensional radial velocity profile plotted against non-dimensional height for $0 \leqslant Q \leqslant 6100$; $A = 500$.

As Q increases above 500 the velocity components grow more rapidly and the maximum value of $-\overline{W}$ moves slightly towards the free surface of the fluid. The values given for $Q = 6100$ are so large that the linearized theory is possibly no longer valid. It appears that for a certain value of Q_s, of order 6×10^3, the temperature difference $T_{(m)} - T_{(2)}$ becomes sufficiently large to cause instability.

6. Conclusions

In agreement with Lance (1958) it is seen that the presence of a positive linear lapse rate of basic temperature decreases the forced velocity components.

However, the inclusion of a constant internal heating term into this problem tends to increase the velocity components.

This internal heating gives rise to a parabolic basic temperature profile, and for values of A and Q such that $A < Q$ there is some region of the fluid which is at a higher temperature than the upper bounding surface. Such a situation is potentially unstable, and the results for a given A show that the forced velocity amplitudes grow as Q increases. The assumption that departures from the basic states are small, enabling the equations to be linearized, restricts the theory to the case in which the basic state is stable. However, these large velocity amplitudes give an indication that instability occurs for a certain value of $Q = Q_s$, where $Q_s > A$.

This agrees with the results of Sparrow *et al.* (1964*a*) who show that the presence of internal heat generation tends to destabilize the fluid. It is difficult to draw further quantitative analogies with this work as the boundary conditions used by Sparrow *et al.* for their homogeneous problem differ from those formulated in (25). It would therefore be valuable to consider the effect of different boundary conditions on the onset of instability, particularly the case of one rigid and one free bounding surface as relevant to this problem.

At this stage the linearization approximation can be examined; for example, without this approximation equation (13) would read

$$-\rho_s(u\,\partial w/\partial r + w\,\partial w/\partial z) - \partial p/\partial z + g\alpha\tau + \nu\rho_s\nabla^2 w = 0.$$

As the fluid is shallow the last term is approximately $\nu\rho_s\partial^2 w/\partial z^2$ and, comparing this with the two non-linear terms in the equation, it is found that the linearization is justified if H is sufficiently small. This linearized theory can thus be expected to yield accurate results when the basic flow is stable, and when the temperature difference between the circumference and centre is sufficiently small.

In order to apply this theory to the Earth's atmosphere, where Q simulates the radiative heat loss, rotation should be included in the model. This would have the effect of introducing a non-zero zonal velocity component, and it is expected that the amplitudes of the other velocity components would be decreased.

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